Zeros of random real sections: Law of Large Numbers and Central Limit Theorem

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Kostlan polynomials

Kostlan polynomials

A Kostlan polynomial of degree d is a random homogeneous polynomial

$$P = \sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha$$

in $\mathbb{R}_d^{\text{hom}}[X_0, \dots, X_n]$, where the $(a_\alpha)_{|\alpha|=d}$ are i.i.d. (independent identically distributed) standard Gaussian variables in $\mathbb{R}$.

For all $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$:

- $|\alpha| = \alpha_0 + \dots + \alpha_n$ is the length of α;
- $\alpha! = \alpha_0! \cdots \alpha_n!$ and, if $|\alpha| = d$, $\binom{d}{\alpha} = \frac{d!}{\alpha!}$;
- $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$.

Reminder on Gaussian distributions

$(V, \langle \cdot\,, \cdot \rangle)$ Euclidean space of dimension N, Λ self-adjoint positive operator.

Definition

A random vector X in V is a centered Gaussian of variance Λ if its distribution admits the density:

$$\frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{\det(\Lambda)}} \exp\left(-\frac{1}{2}\left\langle \Lambda^{-1}x\,, x\right\rangle\right)$$

with respect to the Lebesgue measure. Denoted by $X \sim \mathcal{N}(\Lambda)$.

A standard Gaussian is $X \sim \mathcal{N}(\mathrm{Id})$.

In an orthonormal basis $(e_1, \dots, e_N)$, we have $X = \sum_{i=1}^{N} a_i e_i$, where the coefficients (a_i) are i.i.d. $\mathcal{N}(1)$.

Back to Kostlan polynomials

A Kostlan polynomial is a standard Gaussian P in $\mathbb{R}_d^{\text{hom}}[X_0, \ldots, X_n]$, for the inner product such that $\left\{ \sqrt{\binom{d}{\alpha}} X^\alpha \;\middle|\; |\alpha| = d \right\}$ is orthonormal:

$$\langle P \,, Q \rangle = \frac{1}{\pi^{n+1} d!} \int_{\mathbb{C}^{n+1}} P(z)\overline{Q(z)} e^{-\|z\|^2} \, \mathrm{d}z.$$

Up to a multiplicative constant, it is the only inner product such that:

- the monomials are orthogonal;
- for all $O \in O_{n+1}(\mathbb{R})$, $\langle P \circ O \,, Q \circ O \rangle = \langle P \,, Q \rangle$.

Zeros of Kostlan polynomials

Let $d \geqslant 1$, let $n \geqslant 1$ and let $r \in \{1, \ldots, n\}$.

$$Z_d = P_1^{-1}(0) \cap \cdots \cap P_r^{-1}(0) \subset \mathbb{RP}^n,$$

where $P_1, \ldots, P_r$ are i.i.d. Kostlan polynomials in $\mathbb{R}_d^{\mathrm{hom}}[X_0, \ldots, X_n]$.

Lemma

Almost surely, Z_d is a smooth closed submanifold of $\mathbb{RP}^n$ of codimension r.

Theorem (Kostlan, 1993)

For all $d \geqslant 1$, $\mathbb{E}[\mathrm{Vol}\,(Z_d)] = d^{\frac{r}{2}}$.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

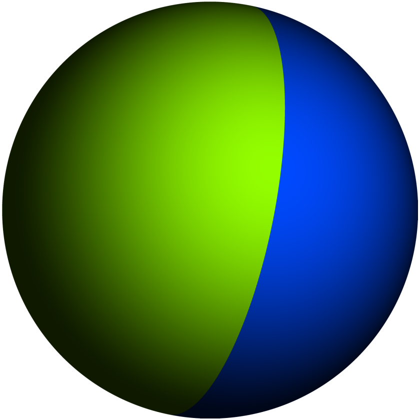

Figure: $d = \quad 1$

Pictures by Vincent Beffara.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

Figure: $d = \quad 2$

Pictures by Vincent Beffara.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

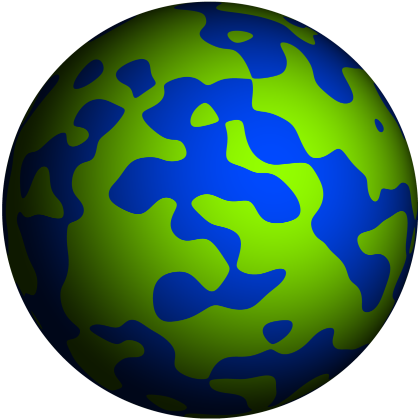

Figure: $d =$ 100

Pictures by Vincent Beffara.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

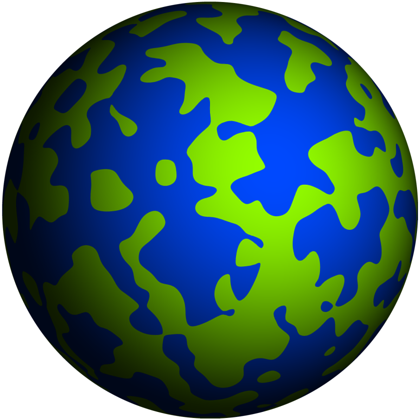

Figure: $d = $ 200

Pictures by Vincent Beffara.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

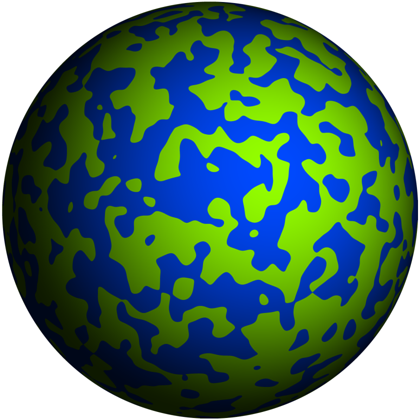

Figure: $d = 500$

Pictures by Vincent Beffara.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

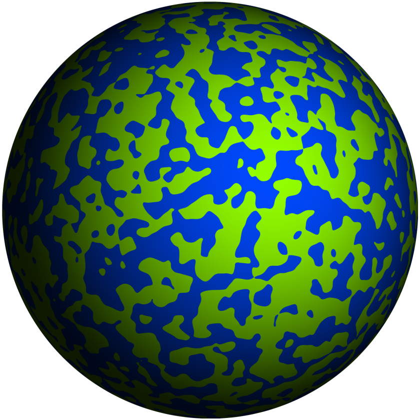

Figure: $d =$ 1000

Pictures by Vincent Beffara.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

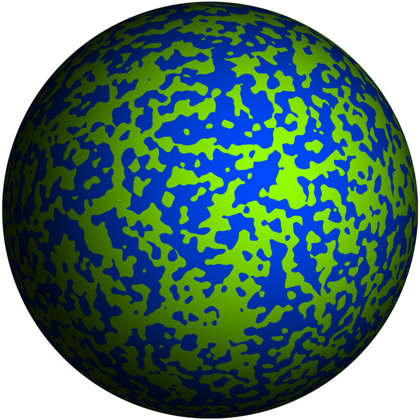

Figure: $d = 2000$

Pictures by Vincent Beffara.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

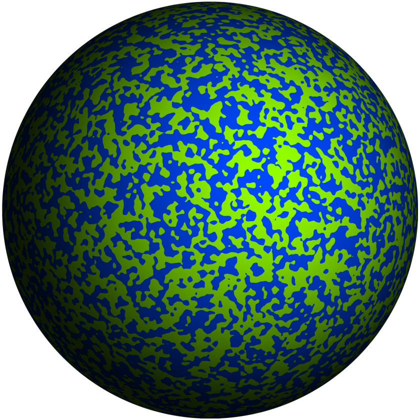

Figure: $d =$ 5000

Pictures by Vincent Beffara.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

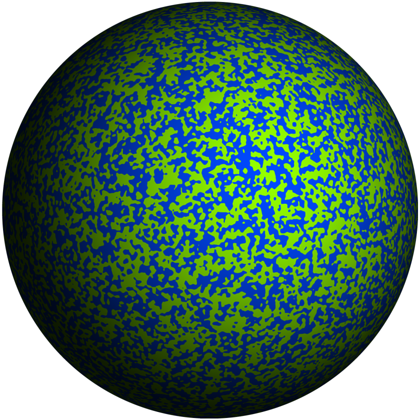

Figure: $d = 10000$

Pictures by Vincent Beffara.

Random algebraic curves in $\mathbb{RP}^2$ ($n = 2$, $r = 1$)

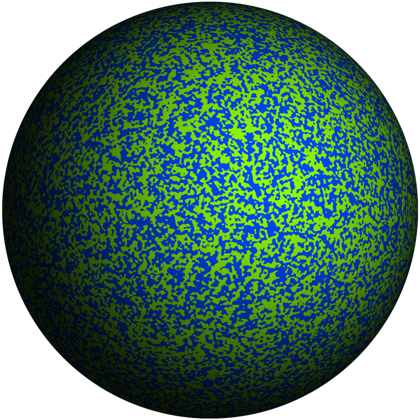

Figure: $d = 20000$

Pictures by Vincent Beffara.

Central Limit Theorem for Kostlan polynomials

Theorem (Dalmao, 2015; Armentano–Azaïs–Dalmao–Leòn, 2018)

There exists $\sigma_{n,r} > 0$ such that, as $d \to +\infty$:

$$\mathrm{Var}(\mathrm{Vol}\left(Z_d\right)) \sim d^{r-\frac{n}{2}} \sigma_{n,r}^2.$$

Moreover, the following holds in distribution:

$$\frac{\mathrm{Vol}\left(Z_d\right) - d^{\frac{r}{2}}}{d^{\frac{r}{2}-\frac{n}{4}} \sigma_{n,r}} \xrightarrow[d \to +\infty]{} \mathcal{N}(1).$$

Random real algebraic submanifolds

Geometric setting

$\mathcal{X}$ complex projective manifold of dimension $n \geqslant 1$,

$(\mathcal{E}, h_{\mathcal{E}})$ rank r Hermitian bundle over $\mathcal{X}$ $(1 \leqslant r \leqslant n)$,

$(\mathcal{L}, h_{\mathcal{L}})$ positive Hermitian line bundle over $\mathcal{X}$,

ω Kähler form induced by the curvature of $\mathcal{L}$.

Assume that $\mathcal{X}$, $\mathcal{L}$ and $\mathcal{E}$ are equipped with compatible real structures (i.e. anti-holomorphic involutions).

Local model for $(\mathcal{E}, h_{\mathcal{E}})$

Take $\mathbb{C}^r \times \mathbb{C}^n \to \mathbb{C}^n$ with the usual complex conjugation.
For all $z \in \mathbb{C}^n$, $h_{\mathcal{E}}(z) = \langle \cdot\, , \cdot \rangle$.

Space of real sections

For any $d \geqslant 1$, $\mathcal{E} \otimes \mathcal{L}^d$ real Hermitian bundle over $\mathcal{X}$, with $h_d = h_{\mathcal{E}} \otimes h_{\mathcal{L}}^d$.

$H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ space of global holomorphic sections and:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \left\{ s \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \;\middle|\; c_d \circ s = s \circ c_{\mathcal{X}} \right\}.$$

We have $N_d = \dim \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) < +\infty$ and $N_d \xrightarrow[d \to +\infty]{} +\infty$.

L^2-inner product

For all s and $s' \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we define:

$$\left\langle s \,,\, s' \right\rangle = \int_{\mathcal{X}} h_d(s, s') \frac{\omega^n}{n!}.$$

Random real sections

We denote $M = \mathrm{Fix}(c_{\mathcal{X}})$ the real locus of $\mathcal{X}$ and assume $M \neq \emptyset$.

For all $d \geqslant 1$, $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, and $Z_d = s_d^{-1}(0) \cap M$.

Lemma

Z_d is almost surely a smooth closed submanifold of M of codimension r, for all d large enough.

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Example

Take $\mathcal{X} = \mathbb{CP}^n$, $\mathcal{L} = \mathcal{O}(1)$ and $\mathcal{E} = \mathbb{C}^r \times \mathcal{X}$ trivial, with their canonical real and metric structures.

Then we have: $M = \mathbb{RP}^n$ and $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \left(\mathbb{R}_d^{\mathrm{hom}}[X_0, \dots, X_n]\right)^r$.
s_d is a r-tuple of independent Kostlan polynomials.

Linear statistics

The Kähler form ω defines a Riemannian metric g on $\mathcal{X}$, hence on M. Volume measures on submanifolds of M are the ones induced by g.

Z_d defines a random Radon measure by:

$$\forall \phi \in \mathcal{C}^0(M), \qquad \langle Z_d \,, \phi \rangle = \int_{x \in Z_d} \phi(x) \left| \mathrm{d}V_{Z_d} \right| .$$

For $\phi = \mathbf{1}$, we have $\langle Z_d \,, \mathbf{1} \rangle = \mathrm{Vol}\,(Z_d)$.

Case of maximal codimension ($r = n$)

Z_d is almost surely finite. The corresponding measure is $\sum_{x \in Z_d} \delta_x$, that is:

$$\langle Z_d \,, \phi \rangle = \sum_{x \in Z_d} \phi(x).$$

Moments, Law of Large Numbers and Central Limit Theorem

Expectation

Let $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and let Z_d denote its real zero set.

Theorem (Gayet–Welschinger, 2015; L., 2016)

For all $\phi \in \mathcal{C}^0(M)$, we have:

$$\mathbb{E}[\langle Z_d\,, \phi\rangle] = d^{\frac{r}{2}} \left(\int_M \phi\,|\mathrm{d}V_M|\right) \frac{\mathrm{Vol}\,(\mathbb{RP}^{n-r})}{\mathrm{Vol}\,(\mathbb{RP}^n)} + \|\phi\|_\infty\, O\Big(d^{\frac{r}{2}-1}\Big)\,.$$

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Corollary (Equidistribution in the mean)

As continuous linear maps on $(\mathcal{C}^0(M), \|\cdot\|_\infty)$,

$$d^{-\frac{r}{2}} \mathbb{E}[Z_d] \xrightarrow[d \to +\infty]{} \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} |\mathrm{d}V_M| \,.$$

Variance

Theorem (L.–Puchol, 2017)

There exists $\sigma_{n,r} > 0$ *such that, for all* $\phi \in \mathcal{C}^0(M)$,

$$\operatorname{Var}(\langle Z_d \,, \phi \rangle) = d^{r-\frac{n}{2}} \sigma_{n,r}^2 \left(\int_M \phi^2 \left| \mathrm{d} V_M \right| \right) + o\left(d^{r-\frac{n}{2}} \right).$$

- $\sigma_{n,r}$ is explicit and only depends on n and r, not on $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$.
- $\sigma_{n,r}$ is the same as in the papers of Armentano–Azaïs–Dalmao–Leòn.
- The positivity of $\sigma_{n,r}$ is non-trivial.

Higher central moments

Definition

Let $p \geqslant 2$ and let X be an L^p real random variable, we denote the p-th central moment of X by: $m_p(X) = \mathbb{E}[(X - \mathbb{E}[X])^p]$.

Definition

For all $p \in \mathbb{N}$, we denote by μ_p the p-th moment of a $\mathcal{N}(1)$ real variable. We have $\mu_{2p} = \frac{(2p)!}{2^p p!}$ and $\mu_{2p+1} = 0$ for all $p \in \mathbb{N}$.

Moments asymptotics

Moments Conjecture

For all $p \geqslant 2$, for all $\phi \in \mathcal{C}^0(M)$, we have:

$$\begin{aligned} m_p(\langle Z_d \,, \phi \rangle) &= \mu_p \operatorname{Var}(\langle Z_d \,, \phi \rangle)^{\frac{p}{2}} + o\left(d^{\frac{p}{2}(r-\frac{n}{2})}\right) \\ &= \mu_p d^{\frac{p}{2}(r-\frac{n}{2})} \sigma_{n,r}^p \left(\int_M \phi^2 \left| \mathrm{d}V_M \right| \right)^{\frac{p}{2}} + o\left(d^{\frac{p}{2}(r-\frac{n}{2})}\right). \end{aligned}$$

Moments asymptotics

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\begin{aligned}
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&= \mu_p d^{\frac{p}{2}(r-\frac{n}{2})} \sigma_{n,r}^p \left(\int_M \phi^2 \left|\mathrm{d}V_M\right|\right)^{\frac{p}{2}} + o\left(d^{\frac{p}{2}(r-\frac{n}{2})}\right).
\end{aligned}
$$

Theorem (Ancona–L., 2019)

The conjecture is true in the case $n = r = 1$.

Strong Law of Large Numbers

We consider a random sequence $(s_d)_{d\geqslant 1}$ of random real sections such that:

- the terms are globally independent,
- for all $d \geqslant 1$, $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.

$(Z_d)_{d\geqslant 1}$ the random sequence of their real vanishing loci.

Theorem (L.–Puchol, 2017; Ancona–L., 2019)

If $n = 1$ or $n \geqslant 3$ then, almost surely, for all $\phi \in \mathcal{C}^0(M)$ we have:

$$d^{-\frac{r}{2}} \langle Z_d\,, \phi \rangle \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \int_M \phi \left|\mathrm{d}V_M\right| .$$

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- When $n \geqslant 3$, Corollary of the variance estimate.
- When $n = 1$, Corollary of the moment estimate for $p = 6$.
- For $n = 2$, it would be implied by the Moments Conjecture for $p = 4$.

Central Limit Theorem

Theorem (Ancona–L., 2019)

If $n = 1$ then, for all $\phi \in \mathcal{C}^0(M) \setminus \{0\}$, the following holds in distribution:

$$\frac{\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]}{\operatorname{Var}(\langle Z_d\,,\phi\rangle)^{\frac{1}{2}}} \xrightarrow[d\to+\infty]{} \mathcal{N}(1).$$

In particular,

$$\frac{1}{d^{\frac{1}{4}}\sigma_{1,1}}\left(\langle Z_d\,,\phi\rangle - d^{\frac{1}{2}}\frac{1}{\pi}\int_M \phi\,|\mathrm{d}V_M|\right) \xrightarrow[d\to+\infty]{} \mathcal{N}\left(\int_M \phi^2\,|\mathrm{d}V_M|\right).$$

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- The Central Limit Theorem is a corollary of the Moments Conjecture.
- Conjectured to hold for any (n, r).
- Proved for Kostlan polynomials (Armentano–Azaïs–Dalmao–Leòn).

Other corollaries

Let $n \geqslant 1$, let $r \in \{1, \dots, n\}$ and let $p \geqslant 1$.

If the Moments Conjecture holds for m_{2p} in dimension n and codimension r, then we have the following corollaries. (True if $p = 1$ or $n = 1$.)

Corollary (Concentration in probability)

Let $(\varepsilon_d)_{d \geqslant 1}$ denote a positive sequence and let $\phi \in \mathcal{C}^0(M)$. Then, as $d \to +\infty$, we have:

$$\mathbb{P}\left(d^{-\frac{r}{2}} \left|\langle Z_d \,, \phi\rangle - \mathbb{E}[\langle Z_d \,, \phi\rangle]\right| > \varepsilon_d\right) = O\left(\left(d^{\frac{n}{4}}\varepsilon_d\right)^{-2p}\right).$$

Corollary (Hole probability)

Let U be a non-empty open subset of M. Then, as $d \to +\infty$, we have:

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O(d^{-\frac{np}{2}}).$$

Proofs of the corollaries

Concentration in probability

By Markov's Inequality for the $2p$-th moment, we have:

$$\mathbb{P}\left(d^{-\frac{r}{2}}\left|\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]\right| > \varepsilon_d\right) \leqslant \varepsilon_d^{-2p} m_{2p}\left(d^{-\frac{r}{2}}\langle Z_d\,,\phi\rangle\right)$$
$$\leqslant \varepsilon_d^{-2p} d^{-pr} m_{2p}\left(\langle Z_d\,,\phi\rangle\right).$$

If the Moments Conjecture holds for m_{2p}, then $m_{2p}\left(\langle Z_d\,,\phi\rangle\right) = O(d^{pr-\frac{pn}{2}})$.

We get a $O\left(\left(d^{\frac{n}{4}}\varepsilon_d\right)^{-2p}\right)$.

Hole probability

Let U be a non-empty open subset of M. There exists $\phi_U \in \mathcal{C}^0(M)$ s.t.:

- for all $x \in U$, $\phi_U(x) > 0$,
- for all $x \in M \setminus U$, $\phi_U(x) = 0$.

Let $\varepsilon > 0$ be such that $\varepsilon < \dfrac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \displaystyle\int_M \phi_U \left|\mathrm{d}V_M\right|$.

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For all d large enough, we have $d^{-\frac{r}{2}}\mathbb{E}[\langle Z_d\,, \phi_U\rangle] > \varepsilon$, so that:

$$\begin{aligned}\mathbb{P}\left(Z_d \cap U = \emptyset\right) &= \mathbb{P}\left(\langle Z_d\,, \phi_U\rangle = 0\right)\\ &\leqslant \mathbb{P}\left(d^{-\frac{r}{2}}\left|\langle Z_d\,, \phi_U\rangle - \mathbb{E}[\langle Z_d\,, \phi_U\rangle]\right| > \varepsilon\right).\end{aligned}$$

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Under the Moments Conjecture for m_{2p}, this is $O((d^{-\frac{n}{4}}\varepsilon)^{-2p}) = O(d^{-\frac{np}{2}})$.

Proof of the Law of Large Numbers (part 1)

Let $p \geqslant 1$. For all $\phi \in \mathcal{C}^0(M)$ we have:

$$\mathbb{E}\left[\sum_{d \geqslant 1}\left(d^{-\frac{r}{2}}\left|\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]\right|\right)^{2p}\right] = \sum_{d \geqslant 1} d^{-pr} m_{2p}(\langle Z_d\,,\phi\rangle).$$

If $m_{2p}(\langle Z_d\,,\phi\rangle) = O(d^{pr-\frac{np}{2}})$ and $np \geqslant 3$, then the sum is finite.

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In this case, a.s., $\displaystyle\sum_{d\geqslant 1}\left(d^{-\frac{r}{2}}\left|\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]\right|\right)^{2p} < +\infty$, hence:

$$d^{-\frac{r}{2}}\langle Z_d\,,\phi\rangle \sim d^{-\frac{r}{2}}\mathbb{E}[\langle Z_d\,,\phi\rangle] \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}(\mathbb{RP}^{n-r})}{\mathrm{Vol}(\mathbb{RP}^{n})}\int_M \phi\,|\mathrm{d}V_M|\,.$$

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If $m_{2p}(\langle Z_d\,,\phi\rangle) = O(d^{pr-\frac{np}{2}})$ and $np \geqslant 3$, then the sum is finite.

In this case, a.s., $\displaystyle\sum_{d \geqslant 1}\left(d^{-\frac{r}{2}}\left|\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]\right|\right)^{2p} < +\infty$, hence:

$$d^{-\frac{r}{2}}\langle Z_d\,,\phi\rangle \sim d^{-\frac{r}{2}}\mathbb{E}[\langle Z_d\,,\phi\rangle] \xrightarrow[d \to +\infty]{} \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)}\int_M \phi\,|\mathrm{d}V_M|\,.$$

- True for $n = 1$ with $p = 3$, and for $n \geqslant 3$ with $p = 1$.
- For $n = 2$, $p = 2$ would be enough.

Proof of the Law of Large Numbers (part 2)

Let $(\phi_k)_{k \geqslant 0}$ be a dense sequence in $(\mathcal{C}^0(M), \|\cdot\|_\infty)$ such that $\phi_0 = \mathbf{1}$.
For all $\phi \in \mathcal{C}^0(M)$ and $k \geqslant 0$,

$$\begin{aligned}\left| d^{-\frac{r}{2}} \langle Z_d \,, \phi \rangle - \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \int_M \phi \left|\mathrm{d} V_M\right| \right| \\ \leqslant \|\phi - \phi_k\|_\infty \left(d^{-\frac{r}{2}} \mathrm{Vol}\left(Z_d\right) + \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \mathrm{Vol}\left(M\right) \right) \\ + \left| d^{-\frac{r}{2}} \langle Z_d \,, \phi_k \rangle - \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \int_M \phi_k \left|\mathrm{d} V_M\right| \right|.\end{aligned}$$

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$$\left| d^{-\frac{r}{2}} \langle Z_d , \phi \rangle - \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \int_M \phi \left|\mathrm{d}V_M\right| \right| \leqslant \|\phi - \phi_k\|_\infty \left(d^{-\frac{r}{2}} \mathrm{Vol}\left(Z_d\right) + \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \mathrm{Vol}\left(M\right) \right) + \left| d^{-\frac{r}{2}} \langle Z_d , \phi_k \rangle - \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \int_M \phi_k \left|\mathrm{d}V_M\right| \right|.$$

A.s., for all $k \geqslant 0$, $d^{-\frac{r}{2}} \langle Z_d , \phi_k \rangle \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \int_M \phi_k \left|\mathrm{d}V_M\right|$.

In particular, the sequence $\left(d^{-\frac{r}{2}} \mathrm{Vol}\left(Z_d\right)\right)_{d\geqslant 1}$ is bounded.

Proof of the Central Limit Thereom

Let $\phi \in \mathcal{C}^0(M) \setminus \{0\}$, for d large enough,

$$X_d = \frac{\langle Z_d \,, \phi\rangle - \mathbb{E}[\langle Z_d \,, \phi\rangle]}{\operatorname{Var}(\langle Z_d \,, \phi\rangle)^{\frac{1}{2}}}$$

is a well-defined, centered and normalized random variable.

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is a well-defined, centered and normalized random variable.

We assume the Moments Conjecture in dimension n and codimension r.

For all $p \geqslant 3$, $m_p(X_d)$ is well-defined for d large enough, and:

$$m_p(X_d) = \frac{m_p(\langle Z_d \,, \phi \rangle)}{\mathrm{Var}(\langle Z_d \,, \phi \rangle)^{\frac{p}{2}}} \xrightarrow[d \to +\infty]{} \mu_p.$$

By the Method of Moments, $X_d \xrightarrow[d \to +\infty]{} \mathcal{N}(1)$ in distribution.

The Method of Moments

By Markov's Inequality, $\mathbb{P}(|X_d| > \varepsilon^{-\frac{1}{2}}) \leqslant \varepsilon$. That is (X_d) is tight.

Tightness implies compactness in distribution.

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Assume $X_{d_n} \xrightarrow[n\to+\infty]{} X$ in distribution.

Since the sequence $\left(\mathbb{E}\left[X_{d_n}^{2p}\right]\right)_{n\geqslant 0}$ is bounded, $(X_{d_n}^p)_{n\geqslant 0}$ is equi-integrable, and:

$$\mathbb{E}[X^p] = \lim_{n\to+\infty} \mathbb{E}\left[X_{d_n}^p\right] = \mu_p.$$

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$$\mathbb{E}[X^p] = \lim_{n\to+\infty} \mathbb{E}\left[X_{d_n}^p\right] = \mu_p.$$

$\mathcal{N}(1)$ is characterized by its moments, hence $X \sim \mathcal{N}(1)$.

Proofs of the moments estimates

The correlation kernel

Correlation function

Recall that a Kostlan polynomial is a random $P \in \mathbb{R}_d^{\text{hom}}[X_0, \dots, X_n]$ s.t.

$$P = \sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha,$$

where the $(a_\alpha)_{|\alpha|=d}$ are i.i.d. $\mathcal{N}(1)$.

P defines a centered Gaussian process $(P(x))_{x \in \mathbb{S}^n}$.

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P defines a centered Gaussian process $(P(x))_{x \in \mathbb{S}^n}$.

As such, characterized by its correlation function: $e_d(x, y) = \mathbb{E}[P(x)P(y)]$.

- $(P(x), P(y))$ is a centered Gaussian $\mathbb{R}^2$ of variance $\begin{pmatrix} e_d(x,x) & e_d(x,y) \\ e_d(y,x) & e_d(y,y) \end{pmatrix}$.
- Taking partial derivatives, we get: $\frac{\partial e_d}{\partial x_i}(x, y) = \mathbb{E}\left[\frac{\partial P}{\partial x_i}(x)P(y)\right]$.

Correlation function of the Kostlan polynomials

$$\begin{aligned} e_d(x,y) &= \mathbb{E}[P(x)P(y)] \\ &= \sum_{|\alpha|=d=|\beta|} \mathbb{E}[a_\alpha a_\beta] \sqrt{\binom{d}{\alpha}}\sqrt{\binom{d}{\beta}} x^\alpha y^\beta \\ &= \sum_{|\alpha|=d} \binom{d}{\alpha} x^\alpha y^\alpha \\ &= (\langle x\,,y\rangle)^d \\ &= \cos\left(\rho(x,y)\right)^d, \end{aligned}$$

where ρ is the geodesic distance in $\mathbb{S}^n$.

Correlation kernel for random real sections

More generally, $s_d \sim \mathcal{N}(\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ defines a centered Gaussian process $(s_d(x))_{x \in \mathcal{X}}$, with values in $\mathcal{E} \otimes \mathcal{L}^d$.

Characterized by its correlation kernel e_d, section of $\left(\mathcal{E} \otimes \mathcal{L}^d\right) \boxtimes \left(\mathcal{E} \otimes \mathcal{L}^d\right)^*$:

$$e_d(x, y) : (\mathcal{E} \otimes \mathcal{L}^d)_y \to (\mathcal{E} \otimes \mathcal{L}^d)_x$$

is the covariance operator of $s_d(x)$ and $s_d(y)$.

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is the covariance operator of $s_d(x)$ and $s_d(y)$.

Lemma

For all $d \geqslant 1$, e_d is the Bergman kernel of $(\mathcal{E} \otimes \mathcal{L}^d, h_d)$.

That is e_d is the integral kernel of the orthogonal projection onto $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ in the space of L^2 sections of $\mathcal{E} \otimes \mathcal{L}^d$.

Estimates for the Bergman kernel

Theorem (Dai–Liu–Ma, 2006)

For all $x \in M$, for all $z \in T_xM$ such that $\|z\| \leqslant d^{\frac{1}{4}}$, we have:

$$e_d\left(x, x + \frac{z}{\sqrt{d}}\right) \simeq e^{-\frac{1}{2}\|z\|^2} I_r$$

as $d \to +\infty$, in the real normal trivialization around x.
This holds in the $\mathcal{C}^k$ sense, uniformly in (x, z).

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as $d \to +\infty$, in the real normal trivialization around x.
This holds in the $\mathcal{C}^k$ sense, uniformly in (x, z).

Theorem (Ma–Marinescu, 2015)

There exists $C > 0$ such that, for all $k \in \mathbb{N}$,

$$\|e_d(x, y)\|_{\mathcal{C}^k} = O\Big(d^{\frac{k}{2}} \exp\left(-C\sqrt{d}\rho(x, y)\right)\Big),$$

as $d \to +\infty$, uniformly in (x, y).

Some heuristic for the expected volume

Cut M into boxes of size $d^{-\frac{1}{2}}$. There are about $d^{\frac{n}{2}}\operatorname{Vol}(M)$ such pieces.

In each box, zooming in at scale $d^{-\frac{1}{2}}$, Z_d converges in distribution.

The expected volume in each box is of order $d^{-\frac{n-r}{2}}$.

The boxes are independent, hence $\mathbb{E}[\operatorname{Vol}(Z_d)]$ is of order $d^{\frac{r}{2}}\operatorname{Vol}(M)$.

Some heuristic for the variance of the volume

Denoting $Z_{d,i} = Z_d \cap (i\text{-th box})$, we have $\mathrm{Vol}(Z_d) = \sum_i \mathrm{Vol}(Z_{d,i})$.

$$\begin{aligned}\mathrm{Var}(\mathrm{Vol}(Z_d)) &= \mathbb{E}\left[\mathrm{Vol}(Z_d)^2\right] - \mathbb{E}[\mathrm{Vol}(Z_d)]^2 \\ &= \sum_{i,j} \mathbb{E}[\mathrm{Vol}(Z_{d,i})\,\mathrm{Vol}(Z_{d,j})] - \mathbb{E}[\mathrm{Vol}(Z_{d,i})]\,\mathbb{E}[\mathrm{Vol}(Z_{d,j})]\end{aligned}$$

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If $i \neq j$, $Z_{d,i}$ and $Z_{d,j}$ are independent. Only terms with $i = j$ contribute.

These terms are of the order of $\mathrm{Vol}(Z_{d,i})^2$, i.e. d^{r-n}.

Hence $\mathrm{Var}(\mathrm{Vol}(Z_d)) \simeq d^{r-\frac{n}{2}}\,\mathrm{Vol}(M)$.

Kac–Rice formulas

Jacobians and evaluation maps

Definition

L linear between Euclidean spaces, its Jacobian is: $\left|\det^{\perp}(L)\right| = \det(LL^*)^{\frac{1}{2}}$.

Note that $\left|\det^{\perp}(L)\right| > 0$ if and only if L is surjective.

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Definition

Let $k \geqslant 1$. For all $d \geqslant 1$ and $x = (x_1, \dots, x_k) \in M^k$, we denote $\mathrm{ev}_x^d : s \mapsto (s(x_1), \dots, s(x_k))$ from $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ to $\bigoplus \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_{x_i}$.

Lemma

Let $k \geqslant 1$, there exists d_k s.t. $\forall d \geqslant d_k$, $\forall x \in M^k \setminus \Delta_k$, ev_x^d is surjective, where:

$$\Delta_k = \left\{(x_1, \dots, x_k) \in M^k \,\middle|\, \exists i \neq j \text{ such that } x_i = x_j\right\}.$$

Kac–Rice formulas

Theorem

Let $k \geqslant 1$*. Let* $d \geqslant d_k$ *and* $s_d \sim \mathcal{N}(\mathrm{Id})$ *in* $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$*. For any continuous* $\Psi : M^k \to \mathbb{R}$*, we have:*

$$\mathbb{E}\left[\int_{Z_d^k \setminus \Delta_k} \Psi(x) \left|\mathrm{d}V_{Z_d}\right|^k\right] = \frac{1}{(2\pi)^{\frac{rk}{2}}} \int_{M^k} \Psi(x) \mathcal{R}_k^d(x) \left|\mathrm{d}V_M\right|^k,$$

where

$$\mathcal{R}_k^d(x) = \frac{\mathbb{E}\left[\prod_{i=1}^k \left|\det^\perp\left(\nabla_{x_i} s_d\right)\right| \,\middle|\, \mathrm{ev}_x^d(s_d) = 0\right]}{\left|\det^\perp\left(\mathrm{ev}_x^d\right)\right|}.$$

Kac–Rice formulas

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- $\mathcal{R}_k^d(x)$ does not depend on ∇.
- If $r = 1$, $\left|\det^\perp\left(\nabla_x s\right)\right| = \|\nabla_x s\|$.

Asymptotic of the expectation for hypersurfaces ($r = 1$)

For all $x \in M$, $\mathrm{ev}_x^d (\mathrm{ev}_x^d)^*$ is the variance of $s_d(x)$, it equals to $e_d(x, x)$.

Kac–Rice for $k = 1$:

$$\mathbb{E}[\langle Z_d \,, \phi \rangle] = \frac{1}{\sqrt{2\pi}} \int_M \phi(x) \frac{\mathbb{E}[\|\nabla_x s_d\| \,|\, s_d(x) = 0]}{\sqrt{e_d(x,x)}} \left|\mathrm{d}V_M\right| .$$

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$(s_d(x), \nabla_x s_d)$ is a centered Gaussian vector of variance:

$$\begin{pmatrix} e_d(x,x) & \partial_{y_j} e_d(x,x) \\ \partial_{x_i} e_d(x,x) & \partial_{x_i}\partial_{y_j} e_d(x,x) \end{pmatrix}_{1\leqslant i,j\leqslant n}.$$

Hence, $\mathcal{R}_d^1(x)$ only depends on the first derivatives of e_d at (x,x).

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Hence, $\mathcal{R}_d^1(x)$ only depends on the first derivatives of e_d at (x,x).

$$\frac{1}{\sqrt{2\pi}} \mathcal{R}_d^1(x) = d^{\frac{1}{2}} \frac{\mathrm{Vol}\left(\mathbb{RP}^{n-1}\right)}{\mathrm{Vol}\left(\mathbb{RP}^{n}\right)} \left(1 + O(d^{-1})\right) .$$

Asymptotics of the central moments

Central moments in terms of non-central moments

Let $p \geqslant 2$ and let $\phi \in \mathcal{C}^0(M)$.

$$\begin{aligned} m_p\left(\langle Z_d\,,\phi\rangle\right) &= \mathbb{E}[(\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle])^p] \\ &= \mathbb{E}\left[\prod_{i=1}^{p}\left(\int_{x_i\in Z_d}\phi(x_i)\,|\mathrm{d}V_{Z_d}| - \mathbb{E}\left[\int_{x_i\in Z_d}\phi(x_i)\,|\mathrm{d}V_{Z_d}|\right]\right)\right] \end{aligned}$$

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$$= \sum_{I\subset\{1,\dots,p\}}(-1)^{p-|I|}\mathbb{E}\left[\int_{Z_d^I}\prod_{i\in I}\phi(x_i)\,|\mathrm{d}V_{Z_d}|^{|I|}\right]\prod_{i\notin I}\mathbb{E}\left[\int_{Z_d}\phi(x_i)\,|\mathrm{d}V_{Z_d}|\right].$$

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For $p = 2$, this is just:

$$m_2(\langle Z_d\,,\phi\rangle) = \mathrm{Var}(\langle Z_d\,,\phi\rangle) = \mathbb{E}\left[\langle Z_d\,,\phi\rangle^2\right] - \mathbb{E}[\langle Z_d\,,\phi\rangle]^2.$$

Maximal codimension is the worst

Recall that, almost surely, Z_d has dimension $n - r$.

If $r < n$, then for all $k \geqslant 2$, for all continuous $\Psi : M^k \to \mathbb{R}$.

$$\int_{Z_d^k} \Psi \left|\mathrm{d}V_{Z_d}\right|^k = \int_{Z_d^k \setminus \Delta_k} \Psi \left|\mathrm{d}V_{Z_d}\right|^k .$$

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If $r = n$, $\left|\mathrm{d}V_{Z_d}\right|^k$ is the counting measure and $Z_d^k \cap \Delta_k$ is not negligible.

$$\int_{Z_d^k} \Psi \left|\mathrm{d}V_{Z_d}\right|^k = \int_{Z_d^k \setminus \Delta_k} \Psi \left|\mathrm{d}V_{Z_d}\right|^k + \begin{pmatrix}\text{other terms tractable} \\ \text{by Kac–Rice}\end{pmatrix}.$$

For $k = 2$,

$$\int_{Z_d^2} \Psi(x,y) \left|\mathrm{d}V_{Z_d}\right|^2 = \int_{Z_d^2 \setminus \Delta_2} \Psi(x,y) \left|\mathrm{d}V_{Z_d}\right|^2 + \int_{Z_d} \Psi(x,x) \left|\mathrm{d}V_{Z_d}\right|.$$

Integral formula for the central moments

If $x = (x_1, \dots, x_p) \in M^p$ and $I \subset \{1, \dots, p\}$, we denote $x_I = (x_i)_{i \in I}$.

Applying Kac–Rice to each term in the expression of $m_p(\langle Z_d\,, \phi \rangle)$,

$$m_p(\langle Z_d\,, \phi \rangle) = \int_{M^p} \left(\prod_{i=1}^{p} \phi(x_i) \right) \mathcal{D}_d^p(x) \, |\mathrm{d}V_M|^p \,,$$

where $$\mathcal{D}_d^p(x) = (2\pi)^{-\frac{pr}{2}} \sum_{I \subset \{1,\dots,p\}} (-1)^{p-|I|} \mathcal{R}_d^{|I|}(x_I) \prod_{i \notin I} \mathcal{R}_d^1(x_i).$$

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where $$\mathcal{D}_d^p(x) = (2\pi)^{-\frac{pr}{2}} \sum_{I \subset \{1,\ldots,p\}} (-1)^{p-|I|} \mathcal{R}_d^{|I|}(x_I) \prod_{i \notin I} \mathcal{R}_d^1(x_i).$$

- The density $\mathcal{D}_d^p$ is singular along Δ_p.
- For $x = (x_1, \ldots, x_p) \in M^p \setminus \Delta_p$, $\mathcal{D}_d^p(x)$ only depends on e_d and its first derivatives at $(x_i, x_j)_{1 \leqslant i,j \leqslant p}$.

Density for $p = 2$

For $p = 2$, we have $\mathcal{D}_d^2(x, y) = (2\pi)^{-r} \left(\mathcal{R}_d^2(x, y) - \mathcal{R}_d^1(x)\mathcal{R}_d^1(y)\right)$.

Lemma

For all d large enough, $\mathcal{D}_d^2$ is integrable over M^2.

Density for $p = 2$

For $p = 2$, we have $\mathcal{D}_d^2(x, y) = (2\pi)^{-r} \left(\mathcal{R}_d^2(x, y) - \mathcal{R}_d^1(x)\mathcal{R}_d^1(y)\right)$.

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Lemma

For all d large enough, uniformly in $x \in M$ and $z \in T_xM$ s.t. $\|z\| \leqslant K \ln d$:

$$\mathcal{D}_d^2\left(x, x + \frac{z}{\sqrt{d}}\right) \simeq d^r \mathcal{D}(\|z\|).$$

Proof of the variance estimate

$$m_2(\langle Z_d\,,\phi\rangle) \simeq \int_{x\in M}\int_{y\in B(x,K\frac{\ln d}{\sqrt{d}})}\phi(x)\phi(y)\mathcal{D}_d^2(x,y)\,|\mathrm{d}V_M|^2$$

$$\simeq d^{-\frac{n}{2}}\int_{x\in M}\left(\int_{\|z\|<K\ln d}\phi(x)\phi\left(x+\tfrac{z}{\sqrt{d}}\right)\mathcal{D}_d\left(x,x+\tfrac{z}{\sqrt{d}}\right)\mathrm{d}z\right)|\mathrm{d}V_M|$$

$$\simeq d^{r-\frac{n}{2}}\left(\int_{x\in M}\phi(x)^2\,|\mathrm{d}V_M|\right)\left(\int_{\mathbb{R}^n}\mathcal{D}(\|z\|)\,\mathrm{d}z\right).$$

Kac–Rice density for $p \geqslant 3$

Conjecture

For all $p \geqslant 3$, the sequence $(d^{-\frac{rp}{2}}\mathcal{D}_d^p)$ is "uniformly integrable" on $M^k \setminus \Delta_k$, up to a good rescaling.

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For all $p \geqslant 3$, there exists $K_p > 0$ s.t., for d large enough, for all $k, l \in \mathbb{N}^*$ with $k + l \leqslant p$, uniformly in $(x_1, \dots, x_k, y_1, \dots, y_l) \in M^{k+l} \setminus \Delta_{k+l}$ such that $\rho(x_i, y_j) \geqslant K_p \frac{\ln d}{\sqrt{d}}$, we have:

$$\mathcal{R}_d^{k+l}(x, y) = \mathcal{R}_d^k(x)\mathcal{R}_d^l(y) + O(d^{\frac{r}{2}(k+l) - \frac{pn}{4} - 1}).$$

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Theorem (Ancona, 2018)

These conjectures are true for $n = r = 1$.

Tearing M^p to pieces

Let $p \geqslant 2$ and let $x = (x_1, \dots, x_p) \in M^p \setminus \Delta_p$.

For all $d \geqslant 1$, we define a graph $G_d(x)$:

- the vertices are $\{1, \dots, p\}$
- there is an edge between i and j iff $i \neq j$ and $\rho(x_i, x_j) \leqslant K_p \frac{\ln d}{\sqrt{d}}$

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The connected components of $G_d(x)$ define a partition $\mathcal{J}_d(x)$ of $\{1, \ldots, p\}$.

That is $\mathcal{J}_d(x) = \{J_1, \ldots, J_m\}$ with $\bigsqcup_{i=1}^m J_i = \{1, \ldots, p\}$.

Let $\mathcal{J}$ be a partition of $\{1, \ldots, p\}$, we define:

$$M^p_{\mathcal{J},d} = \{x \in M^p \setminus \Delta_p \mid \mathcal{J}_d(x) = \mathcal{J}\}.$$

Pieces with a lonely point

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Let $I \subset \{1, \dots, p-1\}$ and let $\widetilde{I} = I \cup \{p\}$. We have:

$$
\begin{aligned}
(-1)^{p-|I|}\mathcal{R}_d^{|I|}(x_I) \prod_{i \notin I} \mathcal{R}_d^1(x_i) &+ (-1)^{p-|\widetilde{I}|}\mathcal{R}_d^{|\widetilde{I}|}(x_{\widetilde{I}}) \prod_{i \notin \widetilde{I}} \mathcal{R}_d^1(x_i) \\
&= (-1)^{p-|I|} \prod_{i \notin \widetilde{I}} \mathcal{R}_d^1(x_i) \left(\mathcal{R}_d^{|I|}(x_I)\mathcal{R}_d^1(x_p) - \mathcal{R}_d^{|I|+1}(x_I, x_p) \right).
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\end{aligned}
$$

For all $x \in M^p_{\mathcal{J},d}$, this term is $O(d^{\frac{p}{2}(r-\frac{n}{2})-1})$.

$$
\mathcal{D}_d^p(x) = \sum_{I \subset \{1,\ldots,p-1\}} \left(\text{ term } I + \text{ term } \widetilde{I} \right) = O(d^{\frac{p}{2}(r-\frac{n}{2})-1}).
$$

Pieces with large clusters

Let $\mathcal{J} = \{J_1, \ldots, J_m\}$ be a partition of $\{1, \ldots, p\}$ without singleton.

We have $2m \leqslant p$.

$$\mathrm{Vol}\left(M^p_{\mathcal{J},d}\right) \simeq \mathrm{Vol}\left(M\right)^m \prod_{i=1}^{m} \left(\frac{\ln d}{\sqrt{d}}\right)^{n(|J_i|-1)}$$

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If $m < \frac{p}{2}$, this volume is $o(d^{-\frac{np}{4}})$. Besides $\mathcal{D}^p_d(x) = O(d^{\frac{rp}{2}})$.

By the uniform integrability condition: $\displaystyle\int_{M^p_{\mathcal{J},d}} \left|\mathcal{D}^p_d\right| |\mathrm{d}V_M|^p = o(d^{\frac{p}{2}(r-\frac{n}{2})})$.

Partitions into pairs

Only remaining possibility: $\mathcal{J} = \left\{ \{a_i, b_i\} \;\middle|\; 1 \leqslant i \leqslant \frac{p}{2} \right\}$ is a partition into pairs of $\{1, \ldots, p\}$.

Partitions into pairs

Only remaining possibility: $\mathcal{J} = \left\{\{a_i, b_i\} \;\middle|\; 1 \leqslant i \leqslant \frac{p}{2}\right\}$ is a partition into pairs of $\{1, \ldots, p\}$.

For such a $\mathcal{J}$, for all $x \in M^p_{\mathcal{J},d}$,

$$\mathcal{D}^p_d(x) = \prod_{i=1}^{\frac{p}{2}} \mathcal{D}^2_d(x_{a_i}, x_{b_i}) + O(d^{\frac{p}{2}(r-\frac{n}{2})-1}).$$

Hence, denoting $\mathcal{K} = \{\{1,2\}\}$,

$$\begin{aligned}\int_{M^p_{\mathcal{J},d}} \left(\prod_{i=1}^{p} \phi(x_i)\right) \mathcal{D}^p_d(x) \left|\mathrm{d}V_M\right|^p &\simeq \prod_{i=1}^{\frac{p}{2}} \int_{M^2_{\mathcal{K},d}} \phi(x_{a_i})\phi(x_{b_i}) \mathcal{D}^2_d(x_{a_i}, x_{b_i}) \left|\mathrm{d}V_M\right|^2 \\ &\simeq \mathrm{Var}(\langle Z_d \,, \phi\rangle)^{\frac{p}{2}} .\end{aligned}$$

Conclusion of the proof

- Partitions with a singleton contribute $o(d^{\frac{p}{2}(r-\frac{n}{2})})$.
- Partitions of cardinality $< \frac{p}{2}$ without singleton contribute $o(d^{\frac{p}{2}(r-\frac{n}{2})})$.
- Partitions into pairs contribute $\mathrm{Var}(\langle Z_d\,, \phi\rangle)^{\frac{p}{2}}$.

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- Partitions into pairs contribute $\mathrm{Var}(\langle Z_d\,,\phi\rangle)^{\frac{p}{2}}$.

The number of partitions into pairs of $\{1,\ldots,p\}$ equals:

$$\mu_p = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ \dfrac{p!}{2^{\frac{p}{2}}\left(\frac{p}{2}\right)!} & \text{if } p \text{ is even.} \end{cases}$$

Thank you for your attention

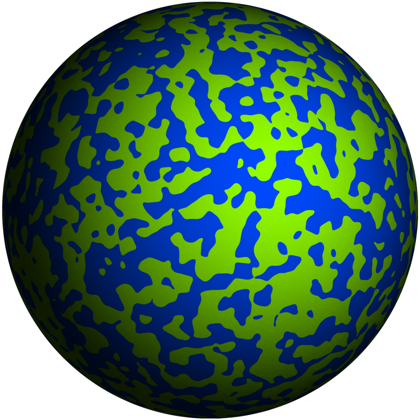

Figure: Random curve in $\mathbb{RP}^2$ of degree $d = 1000$.

Picture by Vincent Beffara.